Defect energy of infinite-component vector spin glasses

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We compute numerically the zero-temperature defect energy ΔE of the vector spin glass in the limit of an infinite number of spin components m, for a range of dimensions $2 \leqslant d \leqslant 5$. Fitting to $\Delta E \sim L^{\theta}$, where L is the system size, we obtain: $\theta \simeq -1.54\,(d=2)$, $\theta \simeq -1.04\,(d=3)$, $\theta \simeq -0.67\,(d=4)$, and $\theta \simeq -0.37\,(d=5)$. These results show that the lower critical dimension d_l (the dimension where θ changes sign) is significantly higher for $m=\infty$ than for finite m (where $2 < d_l < 3$).

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I. INTRODUCTION

There has recently [1–3] been interest in spin glasses wherein the number of spin components, m, is infinite, because this limit provides some simplifications compared with Ising ($m=1$), XY ($m=2$), or Heisenberg ($m=3$) models. For example, in mean field theory (i.e., for the infinite range model) there is no "replica symmetry breaking" [4], so that the ordered state is characterized by a single order parameter q, rather than by an infinite number of order parameters [encapsulated in a function $q(x)$] that are needed [5] for finite m. In addition, there are special numerical techniques [1–3,6,7] which can be used to study finite-range $m=\infty$ spin glasses in which the (finite) sample is solved *exactly* without the statistical errors and equilibration problems inherent in the Monte Carlo methods used for finite m.

There are, however, significant differences between Ising, XY, and Heisenberg spin glasses, on the one hand, and $m=\infty$ spin glasses on the other. In the Ising spin glass in three dimensions there is clearly a finite temperature transition [8], and we have argued [9] that the same is true for XY and Heisenberg spins, although the latter is still somewhat controversial (see, e.g., Refs. [10–12]). Hence, for $m=1$, 2, and 3 the lower critical dimension d_l, below which T_c is zero, is *less than* 3 (in fact $2 < d_l < 3$). However, for $m=\infty$, one finds [3,7] $T_c=0$ in three dimensions, so that d_l must be *greater than* 3 in this case. In fact, Viana [13] makes the surprising claim that $d_l=8$ for $m=\infty$, by attempting to sum up the perturbation expansion. Curiously, the upper critical dimension (above which the critical exponents have mean field values) is *also* predicted [14] to be $d_u=8$, which is again different from the value for finite m where $d_u=6$.

In this paper, we attempt to determine the lower critical dimension of the $m=\infty$ spin glass by computing the zero-temperature "defect energy" ΔE, for a range of dimensions, $2 \leqslant d \leqslant 5$. The defect energy is the characteristic energy change when the boundary conditions are changed from periodic (say) to antiperiodic [15–17]. It is expected that

$$\Delta E \sim L^{\theta}, \tag{1}$$

where L is the system size and θ is a "stiffness exponent." If $\theta > 0$ then the system is stiff on large length scales so that one expects $T_c > 0$, whereas if $\theta < 0$ then it costs very little energy to break up the ground state configuration at large scales, so that presumably $T_c=0$. Hence, d_l is the dimension where $\theta=0$. For the case in which $\theta < 0$, so that $T_c=0$, the correlation length ξ diverges as $T \to 0$ like $\xi \sim T^{-\nu}$, and standard scaling arguments [15,16] then show that $\nu = -1/\theta$.

Our main result is that $\theta < 0$ for the full range of dimensions ($2 \leqslant d \leqslant 5$) that we are able to study, showing that d_l is significantly greater than 5, i.e., much larger than for finite m.

In Sec. II we discuss the models and numerical implementation. In Sec. III we discuss our results from simulation in two to five dimensions. We give our conclusions in Sec. IV.

II. MODEL AND METHOD

We take the Edwards-Anderson [18] Hamiltonian

$$\mathcal{H} = -\sum_{\langle i,j \rangle} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j, \tag{2}$$

where the spins $\mathbf{S}_i$ ($i=1,\ldots,N$) are classical vectors with m components and normalized to length $m^{1/2}$; i.e., $\mathbf{S}_i^2 = m$. The summation is over nearest-neighbor pairs. The interactions J_{ij} connect nearest neighbors and are independent random variables with a Gaussian distribution with zero mean and standard deviation of unity.

At finite temperature and for $m=\infty$ the spin-spin correlation functions

$$C_{ij} \equiv \frac{1}{m} \langle \mathbf{S}_i \cdot \mathbf{S}_j \rangle, \tag{3}$$

are obtained from the following set of equations [1–3,6,7]:

$$T^{-1} C_{ij} = (A^{-1})_{ij}, \tag{4}$$

where

$$A_{ij} = H_i \delta_{ij} - J_{ij}. \tag{5}$$

Here T is the temperature and the H_i ($i=1,\ldots,N=L^d$) are Lagrange multipliers enforcing the normalization of the spins:

$$C_{ii} = 1, \quad (i = 1,2,\ldots,N). \tag{6}$$

To proceed, one solves the N equations [Eq. (6)] to obtain the H_i, and then determines the C_{ij} from Eqs. (4) and (5).

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TABLE I. Estimates of μ and $\theta=-1/\nu$. The values of ν are compared to estimates from finite temperature simulations [3] and a previous calculation [7] at $T=0$.

d	μ	θ	$\nu=-1/\theta$	ν (Ref. [3])	ν (Ref. [7])
2	0.29	−1.54±0.02	0.65±0.01	0.65±0.05	0.65±0.02
3	0.33	−1.04±0.02	0.96±0.02	1.23±0.13	1.01±0.02
4	0.35	−0.67±0.04	1.49±0.09	—	1.5±0.1
5	0.37	−0.37±0.07	2.70±0.51	—	—

At zero temperature, Eqs. (4) and Eq. (5) are no longer well defined. However, since there are no thermal fluctuations, each spin lies parallel to its local field:

$$\mathbf{S}_i = H_i^{-1} \sum_j J_{ij} \mathbf{S}_j. \tag{7}$$

Remarkably, it was shown by Hastings [1] that these local fields are precisely the zero temperature limit of the H_i in Eq. (5). Another interesting result found by Hastings is that the number of independent spin components m_0 used to form the ground state satisfies a bound $m_0 < \sqrt{2N}$. By independent, we mean that we can always define coordinates for the spins such that the projections of the spins are only nonzero for m_0 directions and no spin components are found in remaining $m-m_0$ directions. Furthermore, it is found numerically that

$$[m_0]_{\text{av}} \sim N^{\mu}, \quad (m_0 < m), \tag{8}$$

where $[\cdots]_{\text{av}}$ denotes an average over disorder, and the values of μ we find for dimensions between 2 and 5 are given in Table I.

We now see that to study the $m \to \infty$ limit, we simply need that m should be greater than m_0. Since m_0 is the number of zero eigenvalues of a large matrix $(N \times N)$ for $N=L^d$, it is computationally intensive to determine m_0 for large L and d.

TABLE II. Number of spin components used.

	Number of spin components, m			
L	$d=2$	$d=3$	$d=4$	$d=5$
4	4	6	10	14
5	—	10	14	20
6	—	10	14	27
7	—	10	17	35
8	—	10	19	—
10	6	—	25	—
12	—	14	31	—
16	—	18	—	—
20	8	—	—	—
24	—	25	—	—
32	10	—	—	—
64	14	—	—	—
128	19	—	—	—

TABLE III. Number of samples used in the defect energy calculations.

	Number of samples, N_{samp}			
L	$d=2$	$d=3$	$d=4$	$d=5$
4	1000	1000	1000	1000
5	—	1000	1000	2005
6	—	1000	1000	2098
7	—	1000	1000	1943
8	—	1000	1115	—
10	1000	—	1317	—
12	—	1000	1042	—
16	—	1105	—	—
20	1000	—	—	—
24	—	1792	—	—
32	1000	—	—	—
64	878	—	—	—
128	547	—	—	—

In practice, we can determine $[m_0]_{\text{av}}$ from a smaller range of sizes and fit to Eq. (8). This allows us to extrapolate the value of $[m_0]_{\text{av}}$ to larger sizes. We then choose m to be significantly greater than $[m_0]_{\text{av}}$ for all L. The values of m used in the calculations are shown in Table II. Note some tolerance is required because the precise value of m_0 varies from sample to sample.

To find the ground state, we use a "spin-quench" method [7]. Firstly, the local field on $\mathbf{S}_i$ is computed via

$$H_i = \frac{1}{m^{1/2}} \left| \sum_j J_{ij} \mathbf{S}_j \right|. \tag{9}$$

Next we set $\mathbf{S}_i$ according to Eq. (7). This procedure is applied to each spin of the lattice sequentially, and then iterated to convergence. Our convergence criterion is that the magnitude of the change in each spin is less than about 10^{-7}. For finite m, this method does not guarantee the ground state as

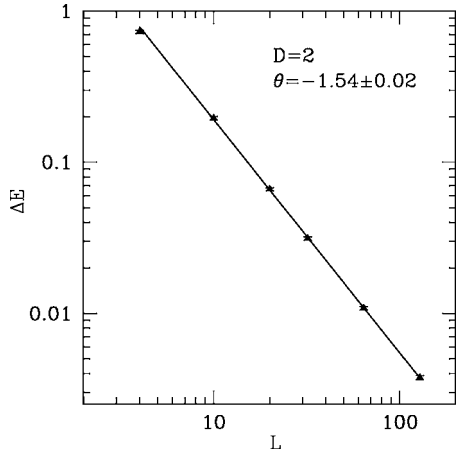

FIG. 1. $d=2$: graph of ΔE against L with $\theta=-1.54$.

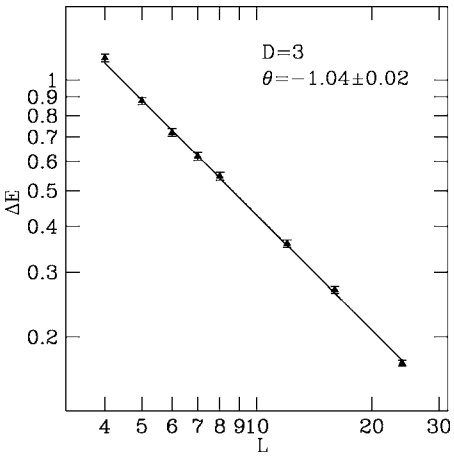

FIG. 2. $d=3$: graph of ΔE against L with $\theta=-1.04$.

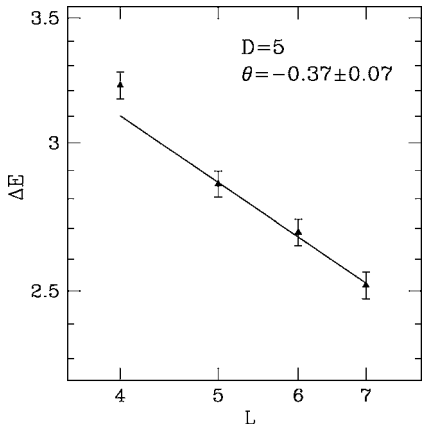

FIG. 4. $d=5$: graph of ΔE against L with $\theta=-0.37$.

there are many solutions to Eq. (7). However, it works in the $m\to\infty$ limit because there is a unique solution [19] in that case. Since there is no change in the ground state for $m > m_0$, the solution is expected to be unique provided this condition is satisfied.

To determine the defect energy ΔE, we first find the ground-state energy with periodic boundary conditions (E_p) for a given set of bonds. Next, we reverse the L^{d-1} bonds that wrap around the system in one direction (x say); i.e., the bonds that connect sites with $x=1$ to those with $x=L$. We then obtain the ground-state energy (E_a) for these "antiperiodic" boundary conditions. On average, neither periodic or antiperiodic boundary conditions is preferred, so that we average the *absolute value* of E_p-E_a over many different configurations of bonds; i.e., the defect energy is defined to be

$$\Delta E = [|E_p - E_a|]_{\mathrm{av}}. \tag{10}$$

We expect that ΔE scales with L according to Eq. (1).

III. RESULTS

We have performed simulations for dimensions $d=2$, 3, 4, and 5. The number of samples for each size and dimension is presented in Table III. The defect energies for different dimensions are plotted in Figs. 1, 2, 3, and 4 together with the fit to Eq. (1). To reduce finite size effects when doing the fits, for $d=2$ we omit the two smallest sizes ($L=4$ and 10), and for higher d we just omit the smallest size ($L=4$).

The exponent θ thus obtained is shown in Table I together with that obtained from previous calculations. The results for $d=2$, 3, and 4 agree well with those of Ref. [7], though we have better statistics than in that work and cover a larger range of sizes (128, 24, and 12 as opposed to 12, 7, and 5). We are not aware of any other results for $d=5$. Comparing with Ref. [3], our results for $d=2$ agree very well, while those for $d=3$ are a little different, at the level of about 2σ, which may reflect some corrections to scaling.

In Fig. 5 we plot our results for θ as a function of d, together with a smooth curve through the points. It is obviously desirable to know the dimension, d_l, where $\theta=0$, but extrapolation of our data to larger d is very uncertain. However, it is clear that d_l must be significantly greater than 5, and hence much greater than the its value for finite m, which is between 2 and 3. It is not possible to test precisely the claim of Viana [13] that $d_l=8$, because we cannot estimate θ for d close to 8. However, our data do not rule out this possibility.

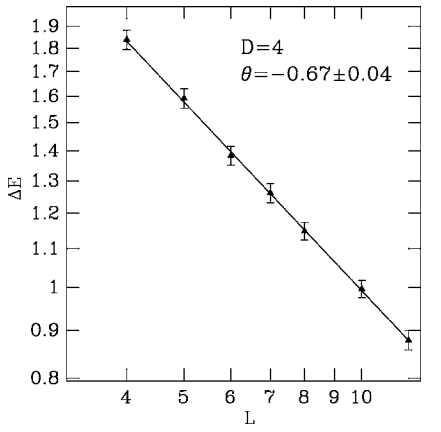

FIG. 3. $d=4$: graph of ΔE against L with $\theta=-0.67$.

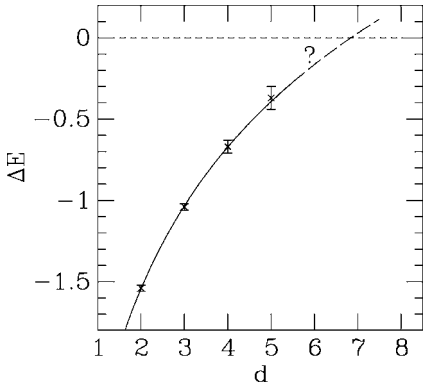

FIG. 5. Graph of θ against d. The solid line is a smooth curve through the data. Its extrapolation to $d>5$ (curve with long dashes) is, however, very uncertain.

IV. CONCLUSIONS

We have computed the zero-temperature stiffness exponent θ for the vector spin glass in the limit where the spins have an infinite number of components. We have obtained better statistics for a larger range of sizes and dimensions that in previous work [7]. Our results for θ agree with those of Ref. [7] for the dimensions ($d=2-4$) considered by them, while the case $d=5$ was not considered there. The trend in our data, shown in Fig. 5, indicates that the lower critical dimension must be significantly larger than 5, the largest dimension we have been able to study, and may equal 8, as predicted by Viana [13], but it is currently not technically possible to determine θ for sufficiently high dimension to test this claim precisely.

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